Einstein Gravity — Supergravity Correspondence

Chiang-Mei Chen

Department of Physics, National Central University, Chungli
E-mail: cmchen@joule.phy.ncu.edu.tw

Dmitri V. Gal'tsov and Sergei A. Sharakin

Department of Theoretical Physics, Moscow State University, Moscow
E-mail: galtsov@grg.phys.msu.su, sharakin@grg1.phys.msu.su

A correspondence between the three-block truncated $11D$ supergravity and the $8D$ pure Einstein gravity with two commuting Killing symmetries is discussed. The Kaluza-Klein two-forms of the $6D$ theory obtained after dimensional reduction along the Killing orbits generate the four-form field of supergravity via an inverse dualization. Thus any solution to the vacuum Einstein equations in eight dimensions depending on six coordinates have $11D$-supergravity counterparts with the non-trivial four-form field. Using this proposed duality we derive a new dyon solution of 11D supergravity describing the $M2$ and $M5$–branes intersecting at a point.

1 Introduction

Recently the embedding of the classical solutions of lower dimensional (truncated) gauged supergravities into the type IIB string and M–theory has attracted much attention as providing a simple way to find solutions of the latter theories in terms of lower dimensional models. This type of embedding is based on the S^n (transversal space of desired solution) compactifications as well as a consistent truncation of scalar fields and vector multiplets. For examples, the charged AdS_5, AdS_4 and AdS_7 black holes had been embedded as the $D3$–branes, $M2$–branes and $M5$–branes respectively. It is worth noting that the charges of the AdS black holes may be interpreted as the angular momenta of the associated branes [1].

Here we suggest another way of embedding [2] of lower-dimensional gravity solutions into eleven-dimensional supergravity using a non-local duality between solutions in the vacuum Einstein theory in eight dimensions and $11D$ supergravity. This symmetry is based on the fact that the six-dimensional reduction of the eight-dimensional vacuum Einstein theory in presence of two spacelike commuting Killing vectors admits, after suitable field redefinitions, the same form as the truncated boson sector of $D = 11$ supergravity compactified on T^5 with some additional restrictions on the metric and the three–form field. This correspondence provides a new scheme of uplifting the $D \leq 8$ dimensional solutions to vacuum Einstein gravity, especially the non-linear superpositions of known Kaluza–Klein solutions such as pp-waves, KK-monopole, Melvin universes etc., to their supergravity counterparts, producing, for instance, intersecting M–fluxbranes [3] as well as a new type composite $M2 \cup M5$–brane [2] corresponding to the Kaluza–Klein dyon black hole.

The proposed new mapping from $8D$ to $11D$ is 'one to many', depending on the choice of two Killing vectors in eight dimensions. This opens a way to proliferate

$11D$ solutions taking as a seed an already known one, lowering it down to eight dimensions according to the parameterization suggested, and then coming back to eleven dimensions with a different choice of (ordered) Killing vectors. This generates the whole class of supergravity solutions related non-locally to each other.

2 11D Supergravity vs. 8D Einstein Gravity

The bosonic sector of $11D$ supergravity contains the eleven-dimensional metric $\hat{g}_{AB}$ and a three-form gauge field $\hat{A}_{[3]}$ described by a Lagrangian which possesses a non-vanishing Chern-Simons term [4]

$$S_{11} = \int d^{11}x\sqrt{-\hat{g}_{11}}\left\{\hat{R}_{11} - \frac{1}{48}\hat{F}^2_{[4]}\right\} - \frac{1}{6}\int \hat{F}_{[4]} \wedge \hat{F}_{[4]} \wedge \hat{A}_{[3]}, \qquad (1)$$

where $\hat{F}_{[4]} = d\hat{A}_{[3]}$. Assuming the space–times can be decomposed into three–block as $M^{11} = M^2(z^a) \times M^3(y^i) \times M^{1,5}(x^\mu)$, where M^2 and M^3 are conformally flat euclidean spaces, $M^{1,5}$ is the pseudoriemannian six-dimensional space–time and, equivalently, the metric has the following form

$$ds^2_{11} = g_2^{\frac{1}{2}}\delta_{ab}dz^a dz^b + g_3^{\frac{1}{3}}\delta_{ij}dy^i dy^j + g_2^{-\frac{1}{4}}g_3^{-\frac{1}{4}}g_{\mu\nu}dx^\mu dx^\nu. \qquad (2)$$

Here the scalars g_2, g_3 and the six-dimensional metric $g_{\mu\nu}$ are functions of x^μ. The corresponding consistent truncation of $11D$ supergravity involves the following ansätz for the $11D$ four–form field $\hat{F}_{[4]}$:

$$\begin{aligned}&\hat{F}_{[4]\mu\nu ab} = \epsilon_{ab}F_{[2]\mu\nu}, \quad \hat{F}_{[4]\mu ijk} = \epsilon_{ijk}\partial_\mu\kappa,\\ &\hat{F}_{[4]\mu\nu\lambda\tau} = H_{[4]\mu\nu\lambda\tau},\end{aligned} \qquad (3)$$

where $H_{[4]} = dB_{[3]}$, $F_{[2]} = dA_{1]}$ and κ are the *six–dimensional* four–form, two–forms and pseudoscalar respectively.

It is straightforward to show that the equations of motion for the six–dimensional variables can be derived from the following action

$$\begin{aligned}S_6 = \int \sqrt{-g_6}\, d^6x\Big\{&R_6 - \frac{1}{2}e^{2\phi}(\partial\kappa)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{3}(\partial\psi)^2\\ &- \frac{1}{4}e^{-\phi}\left(e^{\psi}F^2_{[2]} + \frac{1}{12}e^{-\psi}H^2_{[4]}\right)\Big\} + \int \kappa F_{[2]} \wedge H_{[4]},\end{aligned} \qquad (4)$$

where

$$\phi = -\frac{1}{2}\ln g_3, \quad \psi = -\frac{3}{4}\ln g_2 - \frac{1}{4}\ln g_3. \qquad (5)$$

Now let us start with the $8D$ Einstein theory of gravity

$$S_8 = \int d^8x\sqrt{-g_8}R_8, \qquad (6)$$

on space–times with two spacelike Killing symmetries. In this case the $8D$ metric can be presented as standard Kaluza–Klein pattern

$$ds^2_8 = h_{ab}(d\zeta^a + \mathcal{A}^a_\mu dx^\mu)(d\zeta^b + \mathcal{A}^b_\nu dx^\nu) + (\det h)^{-\frac{1}{4}}g_{\mu\nu}dx^\mu dx^\nu, \qquad (7)$$

where h_{ab}, 2×2 matrix, and $\mathcal{A}^a_\mu$ are two Kaluza-Klein (KK) vectors depending on x^μ only. Under the KK reduction to six dimensions one gets two KK two–forms, and three scalar moduli parameterizing the metric h_{ab}

$$\begin{aligned} h_{ab} &= e^{\frac{2}{3}\psi}\begin{pmatrix} e^{-\phi}+\kappa^2 e^{\phi} & \kappa e^{\phi} \\ \kappa e^{\phi} & e^{\phi}\end{pmatrix}, \\ F_{[2]} &= d\mathcal{A}^1, \quad H_{[2]} = d\mathcal{A}^2, \end{aligned} \tag{8}$$

forming the six-dimensional effective action

$$\begin{aligned} S_6 = \int d^6x\sqrt{-g_6}\Bigg\{ & R_6 - \frac{1}{2}e^{2\phi}(\partial\kappa)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{3}(\partial\psi)^2 \\ & - \frac{1}{4}e^{\psi}\left[e^{-\phi}F_{[2]}^2 + e^{\phi}(H_{[2]}+\kappa F_{[2]})^2\right]\Bigg\}. \end{aligned} \tag{9}$$

Note that the four–form $H_{[4]}$ from the $11D$ supergravity is generated from the KK two–form by inverse dualization

$$H_{[4]\alpha_1\cdots\alpha_4} = \frac{1}{2}\sqrt{-g}e^{\psi+\phi}\epsilon_{\alpha_1\cdots\alpha_4\mu\nu}\left(H_{[2]}^{\mu\nu}+\kappa F_{[2]}^{\mu\nu}\right), \tag{10}$$

after that the effective actions (4) and (9) are identical. Therefore, any $8D$ vacuum solution with at least two spacelike Killing vectors can be embedded into $11D$ supergravity by the following procedure:

1. Writing an $8D$ Ricci flat space with two commuting Killing vectors to the form of (7) and making the identifications of variables by (8).
2. Obtaining the four–form $H_{[4]}$ via the dualization (10).
3. Recovering the $11D$ metric as (2) through inverse variable identification (5).
4. Deriving the $11D$ four–form field $\hat{F}_{[4]}$ by (3).

Possible permutation of two Killing vectors in (7) will lead to different $11D$ solutions. Moreover, if the actual number of commuting spacelike isometries of the $8D$ solution is greater than two, one can generate several different $11D$ solutions via different choice of the ordered pair of ζ^a–directions in (7).

3 $M2$– and $M5$–Brane Solutions

The simplest examples of application of the new embedding procedure are the non–rotating black $M2$–brane and $M5$–brane. Consider the $8D$ pp-wave metric

$$\begin{aligned} ds_8^2 &= H\left(d\zeta_1 + A_t dt\right)^2 + d\zeta_2^2 - H^{-1}f dt^2 \\ &\quad + dx_1^2 + \cdots + dx_k^2 + f^{-1}dr^2 + r^2 d\Omega_{4-k}, \end{aligned} \tag{11}$$

where k in an integer, $(0\le k\le 2)$ and

$$H = 1+\frac{2\delta}{r^{3-k}}, \quad f = 1-\frac{2m}{r^{3-k}}, \quad A_t = \frac{2\sqrt{\delta(m+\delta)}}{r^{3-k}H}. \tag{12}$$

Two commuting Killing vector fields may be chosen either as $(\partial_{\zeta_1}, \partial_{\zeta_2})$ or as $(\partial_{\zeta_2}, \partial_{\zeta_1})$. In the first case, performing the steps described above, we arrive at the $M2$-brane solution

$$\begin{aligned} ds^2_{M2} &= H^{-\frac{2}{3}} \left(-f dt^2 + dz_1^2 + dz_2^2\right) \\ &+ H^{\frac{1}{3}} \Big(dy_1^2 + dy_2^2 + dy_3^2 + dx_1^2 + \cdots + dx_k^2 + f^{-1} dr^2 + r^2 d\Omega_{4-k}\Big), \end{aligned} \tag{13}$$

and $\hat{A}_{t z_1 z_2} = A_t$. In the second case one gets the $M5$-brane

$$\begin{aligned} ds^2_{M5} &= H^{-\frac{1}{3}} \left(-f dt^2 + dz_1^2 + dz_2^2 + dy_1^2 + dy_2^2 + dy_3^2\right) \\ &+ H^{\frac{2}{3}} \left(dx_1^2 + \cdots + dx_k^2 + f^{-1} dr^2 + r^2 d\Omega_{4-k}\right), \\ \hat{A}_{x_1 x_2 \phi} &= -2\sqrt{\delta(m+\delta)} \cos\theta, \qquad \text{for } k = 2, \\ \hat{A}_{x_1 \phi_1 \phi_2} &= -2\sqrt{\delta(m+\delta)} \cos^2\theta, \qquad \text{for } k = 1, \\ \hat{A}_{\psi \phi_1 \phi_2} &= -2\sqrt{\delta(m+\delta)} \cos^3\theta \sin\psi, \qquad \text{for } k = 0. \end{aligned}$$

In a similar way one can obtain rotating branes, solutions endowed with waves and KK monopoles as well as some their intersections.

The above construction illustrates that our mapping from $8D$ to $11D$ is one to many, *i.e.* an unique $8D$ vacuum solution can have several $11D$ supergravity counterparts depending on the choice and the order of two Killing vectors in eight dimensions. This provides a simple way to proliferate $11D$ solutions: taking one known solution one can find its $8D$ vacuum partner and then go back to eleven dimensions with a different choice of the Killing vectors involved. For example, two well-known five–dimensional KK solutions, the Brinkmann wave and the KK-monopole, can be combined in a six–dimensional vacuum solution which has six $11D$ counterparts including all pairwise intersections of four basic solutions: $M2$–brane, $M5$–brane, wave and monopole except the $M2 \perp M5$–brane. One can also find the NUT–generalizations of the rotating M-branes in a way similar to that used in the five-dimensional theory [5].

4 $M2 \cup M5$–Brane Solution

A new interesting solution representing a non-standard intersection of the $M2$ and $M5$ branes at a point (not over a string as demanded by the usual intersection rules) can be obtained starting with the $5D$ Kaluza–Klein dyon. The most general dyonic black hole solution of KK theory was found by Gibbons and Wiltshire [6] and its rotating version by Clément [7], Rasheed [8] and Larsen [9]. In the static case this is a three–parameter family (with mass, electric and magnetic charges m, q, p):

$$\begin{aligned} ds^2_8 &= \frac{B}{A} \left(d\zeta_1 + \mathcal{A}_\mu dx^\mu\right)^2 + d\zeta_2^2 + dx_1^2 + dx_2^2 \\ &+ \sqrt{\frac{A}{B}} \left\{ \frac{-\Delta}{\sqrt{AB}} dt^2 + \sqrt{AB} \left(\frac{dr^2}{\Delta} + d\Omega_2\right) \right\}, \end{aligned} \tag{14}$$

where

$$
\begin{aligned}
A &= \left(r - \frac{d}{\sqrt{3}}\right)^2 - \frac{2dp^2}{d - \sqrt{3}m}, \\
B &= \left(r + \frac{d}{\sqrt{3}}\right)^2 - \frac{2dq^2}{d + \sqrt{3}m}, \\
\Delta &= r^2 - 2mr + p^2 + q^2 - d^2,
\end{aligned} \tag{15}
$$

and the KK vector field is given by

$$
\mathcal{A}_t = \frac{C}{B}, \qquad \mathcal{A}_\phi = 2p\cos\theta, \tag{16}
$$

with

$$
C = 2q\left(r - \frac{d}{\sqrt{3}}\right). \tag{17}
$$

The scalar charge d satisfies the cubic equation

$$
\frac{q^2}{d + \sqrt{3}m} + \frac{p^2}{d - \sqrt{3}m} = \frac{2}{3}d \tag{18}
$$

Using the embedding procedure, one obtains the $11D$ solution

$$
\begin{aligned}
ds_{11}^2 &= \left(\frac{B}{A}\right)^{-\frac{1}{6}} \left\{ \frac{-\Delta}{\sqrt{AB}}dt^2 + \sqrt{AB}\left(\frac{dr^2}{\Delta} + d\Omega_2\right) \right\} \\
&\quad + \left(\frac{B}{A}\right)^{-\frac{2}{3}} \left(dz_1^2 + dz_2^2\right) + \left(\frac{B}{A}\right)^{\frac{1}{3}} \left(dy_1^2 + dy_2^2 + dy_3^2 + dx_1^2 + dx_2^2\right), \\
\hat{A}_{tz_1z_2} &= \mathcal{A}_t, \qquad \hat{A}_{\phi z_1 z_2} = \mathcal{A}_\phi.
\end{aligned} \tag{19}
$$

This solution for $p = 0$ reduces to the usual $M2$–brane while for $q = 0$ — to the $M5$–brane (localized only on a part of the full transverse space). However, for both q, p non-zero, this is a new dyonic brane: contrary to the $M2 \subset M5$–brane [10,11,12], for which the electric brane lies totally within the magnetic one, and contrary to the intersecting $M2 \perp M5$–brane (with a common string), now the common part of the world-volumes is zero-dimensional (we suggest the name $M2 \cup M5$).

$$
\begin{array}{lllllllll}
M2: & z_1 & z_2 & & & & & \\
M5: & & & y_1 & y_2 & y_3 & x_1 & x_2
\end{array}
$$

The electric/magnetic charge densities are $Q = 8\pi q$ and $P = 8\pi p$ respectively.

One can also obtain the rotating version of $M2 \cup M5$–brane (with one rotational parameter) through the following transformation:

$$
\begin{aligned}
dt &\to dt + \omega d\phi, \qquad \Delta \to \Delta + a^2, \\
A &\to A + a^2\cos^2\theta + \frac{2jpq\cos\theta}{(m + \frac{d}{\sqrt{3}})^2 - q^2},
\end{aligned}
$$

$$B \to B + a^2\cos^2\theta - \frac{2jpq\cos\theta}{(m-\frac{d}{\sqrt{3}})^2 - p^2},$$

$$C \to C - \frac{2jp(m+\frac{d}{\sqrt{3}})\cos\theta}{(m-\frac{d}{\sqrt{3}})^2 - p^2}, \tag{20}$$

where

$$\omega = \frac{2j\sin^2\theta}{\Delta - a^2\sin^2\theta}\Big[r - m + \frac{(m+\frac{d}{\sqrt{3}})(m^2+d^2-p^2-q^2)}{(m+\frac{d}{\sqrt{3}})^2 - q^2}\Big]. \tag{21}$$

In terms of new variables one component of the KK vector potential, $\mathcal{A}_t$, remains the same while another becomes

$$\mathcal{A}_\phi = \frac{C}{B}\omega + \frac{2p\Delta\cos\theta}{\Delta - a^2\sin^2\theta} - \frac{2jq\sin^2\theta\Big[r(m-\frac{d}{\sqrt{3}}) + \frac{md}{\sqrt{3}} - p^2 - q^2 + d^2\Big]}{(\Delta - a^2\sin^2\theta)\Big((m+\frac{d}{\sqrt{3}})^2 - q^2\Big)}. \tag{22}$$

The angular momentum of the system, j, is given by

$$j^2 = a^2\frac{\Big[(m+\frac{d}{\sqrt{3}})^2 - q^2\Big]\Big[(m-\frac{d}{\sqrt{3}})^2 - p^2\Big]}{m^2+d^2-p^2-q^2}. \tag{23}$$

5 Conclusion

We have presented a new generating technique to obtain solutions to $11D$ supergravity with non-zero four-form field starting with the solutions of vacuum 8D gravity with two commuting Killing spacelike Killing symmetries. This method can also be combined with other tools to get physically interesting solutions. In particular, a direct application of our procedure does not allow to obtain $11D$ branes *localized* on all coordinates of transverse space, since our basic ansatz (2) is not general enough. However one still can use it to explore possible geometric $11D$ structures and then to try to localize solutions in a straightforward way. Thus we have given a purely vacuum interpretation to a certain class of the $11D$ supergravity M-branes via new non-local duality between $11D$ supergravity and $8D$ vacuum Einstein gravity. It is worth noting that the very natural correspondence between the Killing spinor equations in both theories may have a deeper meaning in the context of $8D$ supergravities [2]. In this connection it has to be emphasized that the type of duality described here is by no means related to the direct dimensional reduction from $11D$ to $8D$ theory. It is also different from the duality between the $8D$ self-dual gauge theories and $10D/11D$ supergravities discussed in [10,11] where the four-form has a non-geometric origin.

Acknowledgments

The work of CMC was supported in part by the National Science Council under grant NSC 89-2112-M-008-016.

References

1. C. Cvetič, M.J. Duff, P. Hoxha, J.T. Liu, H. Lü, J.X. Lu, R. Martinez–Acosta, C.N. Pope, H. Sati and T.A. Tran, *Embedding AdS Black Holes in Ten and Eleven Dimensions*, `hep-th/9903214`.
2. C.-M. Chen, D.V. Galt'sov and S.A. Sharakin, *Vacuum Interpretation for Supergravity M-Branes*, `hep-th/9908133`.
3. C.-M. Chen, D.V. Galt'sov and S.A. Sharakin, *Intersecting M–Fluxbranes*, *Grav. Cosmol.* **5**, 45 (1999); `hep-th/9908132`.
4. K.S. Stelle, *Lectures on Supergravity p–Branes*, `hep-th/9710119`.
5. C.-M. Chen, D.V. Gal'tsov, K. Maeda and S.A. Sharakin, *$SL(4,R)$ Generating Symmetry in Five–Dimensional Gravity Coupled to Dilaton and Three-Form*, *Phys. Lett. B* **453**, 7 (1999); `hep-th/9901130`.
6. G. Gibbons and D. Wiltshire, *Black Holes in Kaluza–Klein Theory*, *Ann. Phys.* **167**, 201 (1986); Erratum: *ibid* **176**, 393 (1987).
7. G. Clément, *Rotating Kaluza–Klein Monopoles and Dyons*, *Phys. Lett. A* **118**, 11 (1986); *Stationary Solutions in Five–Dimensional General Relativity*, *Gen. Rel. Grav.* **18**, 137 (1986).
8. D. Rasheed, *The Rotating Dyonic Black Holes of Kaluza–Klein Theory*, *Nucl. Phys. B* **454**, 379 (1995); `hep-th/9505038`.
9. F. Larsen, *Rotating Kaluza–Klein Black Holes*, `hep-th/9901102`.
10. J.M. Izquierdo, N.D. Lambert, G. Papadopoulos and P.K. Townsend, *Dyonic Membranes*, *Nucl. Phys. B* **460**, 560 (1996); `hep-th/9508177`.
11. M.B. Green, N.D. Lambert, G. Papadopoulos and P.K. Townsend, *Dyonic p-Branes from Self-Dual $(p+1)$-Branes*, *Phys. Lett. B* **384**, 86 (1996); `hep-th/9605146`.
12. M.S. Costa, *Composite M–branes*, *Nucl. Phys. B* **490**, 202 (1997); `hep-th/9609181`; *Black Composite M–branes*, *Nucl. Phys. B* **495**, 195 (1997); `hep-th/9610138`.